REPORT ON "A PRIORI BOUNDS FOR SOME INFINITELY RENORMALIZABLE COMBINATORICS: I. BOUNDED PRIMITIVE COMBINATORICS"

One of the crucial issues in renormalization theory is the proof of "apriori bounds". Broadly, it ensures that renormalization does not diverge, and opens the possibility of taking limits and studying their geometry. This seemingly technical (when precisely formulated for specific classes of maps) type of result is thus the harder analytical step that needs to be taken before softer analytical arguments can be brought in.

Particularly after the work of Sullivan, apriori bounds have been understood as a central step in the proof of universality in renormalization (it is of course also fundamental in the latter works of McMullen and Lyubich). Apriori bounds are also the basic means by which the strongest results about the local connectivity of Julia sets and of the Mandelbrot set are ultimately obtained. It is also determinant in the investigation of measure-theoretical questions: the proof by Buff and Cheritat of the existence of Julia sets with positive Lebesgue measure was enabled by the proof of apriori bounds (by Inou and Shishikura), for some class of dynamical systems. The investigation of other geometric questions, for instance, Hausdorff dimension, also often needs apriori bounds.

Being so important, an enormous effort has been dedicated through the years to the proof of apriori bounds for larger and larger classes of maps, and it is very difficult to bring a significant new contribution to this topic.

This paper proves apriori bounds for an important class of quadratic polynomials, and does so by introducing new perspectives on the issue, which we can hope will have several other ramifications. There is no doubt this work is worth being published in any of the best of mathematical journals.

0.1. **The result.** The class of maps this paper deals with are infinitely renormalizable quadratic-like maps with bounded primitive combinatorics.

A quadratic-like map is just ramified double covering $f : U \to V$ where $U, V \subset \mathbb{C}$ are Jordan domains with $\overline{U} \subset V$ (we assume that the critical point is 0). The Julia set of quadratic-like map is $J(f) = \cap_{k \geq 0} f^{-k}(U)$, it is connected if and only if $f^k(0) \in U$ for every $k \geq 0$. We shall only consider in this discussion quadratic-like maps with connected Julia set, which we will call ql-maps. A quadratic polynomial $z \mapsto z^2 + c$ can be restricted to a ql-map if and only if c belongs to the Mandelbrot set.

A ql-map is said to be primitively renormalizable with period p if there exists a Jordan domain $U' \subset U$ such that $f' = f^p : U' \to f^p(U')$ is (well defined and) a ql-map as well, and moreover $f^i(U) \cap U = \emptyset$, $1 \leq i \leq p-1$. If we can primitively renormalize again and again, with bounded periods, we are in the class considered in this paper, which we will denote by $\mathcal{P}$ from now on.

Starting from such a map, renormalization thus gives a sequence of quadratic-like maps f_n. The domain and range of f_n are ill defined in the above procedure, so it is natural to consider f_n as a germ of an analytic function near its Julia set, and ask how good a quadratic-like extension such germ may possess. One reasonable measure of the goodness of a ql-germ f, denoted by $\mathrm{mod}(f)$, is the suppremum of the modulus $\mathrm{mod}(V_n \setminus U_n)$, taken over all U_n and V_n such that f_n admits a quadratic-like extension $U_n \to V_n$. For every $\epsilon > 0$ the set of ql-maps with $\mathrm{mod}(f) \geq \epsilon$, considered up to rescaling, is compact. Apriori bounds means that $\inf \mathrm{mod}(f_n) > 0$.

As described before, apriori bounds, once established, permit the application of an arsenal of techniques, and thus has many consequences. Two very important ones are that the Julia set of a ql-map of class $\mathcal{P}$ is locally connected, and the Mandelbrot set is locally connected at c whenever the quadratic polynomial $z \mapsto z^2 + c$ restricts to a ql-map in class $\mathcal{P}$.

The class $\mathcal{P}$ is interesting in itself: Feigenbaum-Coullet-Tresser universality, for instance, regards precisely infinitely renormalizable maps with bounded type combinatorics (unfortunately the classical Feigenbaum polynomial escapes the scope of this paper, since it is not *primitively* renormalizable...). However, I believe that the largeness of the class $\mathcal{P}$ considered in this paper is not as important as the fact that it escaped completely from all previous analysis: combination with other approaches will surely lead to improvements soon. In fact, the result obtained in this paper already has been combined with others to prove apriori bounds for a larger class of ql-maps (containing, for instance, infinitely renormalizable maps with arbitrary real primitive combinatorics) in more recent work of Kahn and Lyubich.

0.2. **The presentation.** I will not try to explain in a few words the basic ideas of the paper: in fact the introduction already contains an outline that turns out to be extremely helpful and clear... in retrospect! At least for me, even the basic strategy could only be appreciated after considerable immersion, during an oral presentation by the author. Indeed I had lots of difficulties reading this paper, and I would like to dedicate the rest of my report to this point, and how it could be improved.

(Let me insist that my focus on the improvement of the exposition is motivated by an understanding that, much more than just providing a proof of a very important result, this paper introduces ideas that should become widely known in the field, and I believe it is worth to make an effort to facilitate this process.)

This paper introduces several new concepts, such as the notion of pseudo-quadratic-like maps and canonical weighted arc diagrams. The presentation follows an "axiomatic style", where abstract concepts are introduced and formal properties are deduced in some generality, while the actual concrete objects of interest are kept beyond sight for a long time. I found this approach very difficult to follow for several reasons.

It is difficult to grasp along the paper which formal properties are going to be most fundamental in the proof, so any difficulties the reader encounters (sometimes related to otherwise harmless mistakes) look like potential major obstacles (especially given the knowledge that the result is so delicate).

Arguments which are related to more well known techniques (and even recent, but accepted, ones such as the Covering Lemma of Kahn and Lyubich), which would make the reader feel more at ease, are carried on relatively late. It would be much more comfortable to develop the arguments "to the extent of what can be carried on with current technology" first, making it clear when the new perspective becomes most necessary, and motivating its development.

Let me mention one example of presentation decision which needlessly makes the paper hard to read. The concept of pseudo-quadratic-like map is new and abstract, and it is introduced early. After a big effort to try to get a feeling for it, this concept is immediately forgotten, only to be applied in the very end of the argument.

Anyway, I eventually got in touch with the author, and now I understand the argument and can attest for its correction (the mathematical mistakes I found along my reading were communicated to the author at that moment, and none of them did affect the main argument in the end). The oral presentation of the author followed a different path (almost opposite to the one taken in the paper...), avoiding the shortcomings I perceived. It seemed to me that the written exposition could be modified to achieve the same effect, without any compromise in rigour.

The correctness of this paper not being in doubt, I would still be very happy if the author produced a revised version addressing such issues.